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## Discrete States in the $W_3$ String

C.N. Pope<sup>(1)\*</sup>, E. Sezgin<sup>(1)◇</sup>, K.S. Stelle<sup>(2)\$</sup> and X.J. Wang<sup>(1)</sup>

<sup>(1)</sup> *Center for Theoretical Physics, Texas A&M University,  
College Station, TX 77843-4242, USA.*

<sup>(2)</sup> *The Blackett Laboratory, Imperial College,  
Prince Consort Road, London SW7 2BZ, UK.*

### ABSTRACT

We construct the low-lying discrete states of the two-scalar  $W_3$  string. This includes states that correspond to the analogues of the ground ring generators of the ordinary two-dimensional string. These give rise to infinite towers of discrete states at higher levels.

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## 1. Introduction

A two-dimensional string, having no transverse dimensions, might naively be expected to have no physical content, beyond the tachyon. However, as is now well known, there are in fact discrete excited states at special values of the momenta [1]. These states occur with non-standard ghost numbers as well as the standard one [2,3]. The physical operators associated with these states give rise to an infinite-dimensional algebra of conserved charges [3]. The Ward identity associated with this symmetry provides a powerful tool for determining the correlation functions of the theory.

It is natural to investigate the analogous questions for the  $W_3$  string. There seems to be no notion of a critical dimension for a  $W_3$  string, since a critical realisation of the  $W_3$  algebra in terms of scalar fields always requires background charges, regardless of the number of scalars. To this extent the  $W_3$  string is like the so-called “non-critical string,” namely a string where criticality is achieved by choosing  $d \neq 26$  scalar fields, with background charges. The special feature of  $d = 2$  for ordinary strings is that the gauge symmetries of the excited states remove all transverse degrees of freedom. It is not so clear for the  $W_3$  string what the analogous dimension should be. Results from previous work on  $W$  strings, and the work described here, suggest that it is most natural to study the case of a two-scalar realisation of  $W_3$ . In some sense, as we shall see, this theory exhibits some features of the  $d = 1$  string as well as some features of the  $d = 2$  string.

In this paper, we shall construct some of the low-lying discrete states of the two-scalar  $W_3$  string, including some which exhibit properties analogous to the ground ring discovered in [3]. In order to highlight the similarities and the differences, we shall begin in section 2 by summarising the salient features of the discrete states of the ordinary string. In view of the remark made above, it is appropriate to consider discrete states for the one-dimensional string as well as the two-dimensional string. In section 3 we then describe the construction of discrete states in the  $W_3$  string. In particular, we find analogues of the  $x$  and  $y$  operators of Witten, which are the building blocks of the ground ring. Section 4 contains conclusions and discussion.

## 2. Discrete states in ordinary strings

With the exception of the tachyon, all physical states of the two-dimensional string occur for only discrete values of the momenta. By definition, a physical state is one that is annihilated by the BRST operator  $Q_{BRST}$ , whilst not itself being expressible as  $Q_{BRST}$  of any state. Physical states are characterised by their level number  $\ell$  and their ghost number  $G$ . We use the convention in which  $G = 0$  for the standard ghost vacuum  $|- \rangle \equiv c_1|0 \rangle$ , where  $|0 \rangle$  is the  $SL(2, C)$ -invariant vacuum. A general state is obtained by acting on the tachyon state  $|\vec{p}, - \rangle$  with a string of ghost operators  $c_{-m}$ ,  $m \geq 0$ , antighost operators  $b_{-m}$ ,  $m \geq 1$ ,

and matter operators  $\vec{\alpha}_{-m}$ ,  $m \geq 1$ . Thus one can easily see that the states at level  $\ell$  can have ghost numbers in the interval

$$\left[ \frac{1 - \sqrt{8\ell + 1}}{2} \right] \leq G \leq \left[ \frac{1 + \sqrt{8\ell + 1}}{2} \right], \quad (2.1)$$

where  $[x]$  denotes the integer part of  $x$ .

For the two dimensional string, the physical discrete states at a given level fall into quartets with the following ghost numbers:  $(G, G + 1, G + 1, G + 2)$  and their conjugates with ghost numbers  $(-G + 1, -G, -G, -G - 1)$ . (The conjugate of a state with ghost number  $G$  and momentum  $(p_1, p_2)$  is a state with ghost number  $(-G + 1)$  and momentum  $(-p_1 - 2Q, -p_2)$ .) Starting from the state with ghost number  $G$ , which we shall call the “prime state,” the remaining three states of the quartet can be obtained by acting with  $a_\varphi$ ,  $a_X$  and  $a_\varphi a_X$ , where  $a_\varphi \equiv [Q_{BRST}, \varphi] = c\partial\varphi - \sqrt{2}\partial c$  and  $a_X \equiv [Q_{BRST}, X] = c\partial X$ . Here  $\varphi$  is the Liouville field with background charge  $Q = \sqrt{2}$  and  $X$  is the matter field.<sup>†</sup> The operators  $a_\varphi$  [4] and  $a_X$  are both BRST non-trivial, even though they are formally written as BRST commutators, since neither  $\varphi$  nor  $X$  is a well-defined conformal field. The action of the  $a_\varphi$  and  $a_X$  operators is indicated in Fig. 1 below.

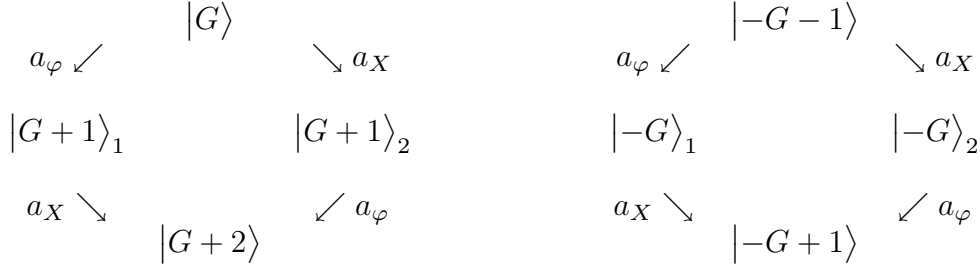


Figure 1

The structure of the states in a quartet and its conjugate, generated from a prime state  $|G\rangle$ .

At level  $\ell = 0$ , we have the tachyon  $|\vec{p}, -\rangle$ , with  $G = 0$ . This is an exception to the above rules, firstly because it has continuous momentum and secondly because it is annihilated by

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<sup>†</sup> An alternative description of the mappings induced by  $a_\varphi$  and  $a_X$  is as follows. Consider the prime state  $|\vec{p}_0, G\rangle$ , with discrete momentum  $\vec{p}_0$ , satisfying  $Q_{BRST}|\vec{p}_0, G\rangle = 0$ . Extrapolating this to arbitrary momentum in the exponential gives a state  $|\vec{p}, G\rangle$  that is annihilated by  $Q_{BRST}$  at  $\vec{p} = \vec{p}_0$ . Since for generic  $\vec{p}$  the state  $Q_{BRST}|\vec{p}, G\rangle$  is manifestly BRST invariant, it follows that the states

$$|\vec{p}, G + 1\rangle_i \equiv \frac{d}{dp_i} Q_{BRST} |\vec{p}, G\rangle \Big|_{\vec{p}=\vec{p}_0}, \quad i = 1, 2$$

are also annihilated by  $Q_{BRST}$ . However they are not BRST trivial, since at  $\vec{p} = \vec{p}_0$  they cannot be written as  $Q_{BRST}$  acting on any states. (This is a restatement of an argument in [3].) They are in fact the same as the states obtained by acting on  $|\vec{p}_0, G\rangle$  with  $a_\varphi$  or  $a_X$ . To see this, note that  $|\vec{p}, G\rangle$  can be written as  $R(b, c, \partial\vec{\varphi})e^{\vec{p}\cdot\vec{\varphi}(0)}|-\rangle$ , and so  $\frac{d}{dp_i} Q_{BRST} |\vec{p}, G\rangle \Big|_{\vec{p}=\vec{p}_0} = [Q_{BRST}, \varphi_i] R(b, c, \partial\vec{\varphi})e^{\vec{p}_0\cdot\vec{\varphi}(0)}|-\rangle$ , where  $\vec{\varphi} = (\varphi, X)$ .

$a_X$ . Thus the generic quartet structure degenerates in this case. Acting with  $a_\varphi$  on  $|\vec{p}, -\rangle$  gives the  $G = 1$  state  $|\vec{p}, +\rangle$ . This is in fact the conjugate of the tachyon  $|\vec{p}', -\rangle$ , with  $\vec{p}' = -\vec{p} - (2Q, 0)$ .

At level  $\ell = 1$ , there is a single discrete state at the lowest ghost number,  $G = -1$ . This is  $b_{-1}|\vec{0}, -\rangle$ ; in other words the  $SL(2, C)$  vacuum  $|0\rangle$ , with zero momentum. Acting with the  $a_\varphi$  and  $a_X$  operators then fills out a quartet, with ghost numbers  $(-1, 0, 0, 1)$ . The conjugates of these states give a quartet with ghost numbers  $(2, 1, 1, 0)$ . The prime state  $|0\rangle$  corresponds to the  $G = 0$  identity operator.<sup>†</sup>

At level  $\ell = 2$ , there are two discrete states at the lowest ghost number,  $G = -1$ . Each of these prime states has its quartet partners and the conjugate quartet. The operators corresponding to these two prime states are the  $x$  and  $y$  ground ring generators of Witten, with  $G = 0$ . They are given by

$$x = \left( b c + \frac{1}{2} Q (\partial\varphi + i\partial X) \right) e^{\frac{1}{2} Q (\varphi - iX)}, \quad (2.2a)$$

$$y = \left( b c + \frac{1}{2} Q (\partial\varphi - i\partial X) \right) e^{\frac{1}{2} Q (\varphi + iX)}, \quad (2.2b)$$

where the background-charge parameter  $Q$  takes the value  $\sqrt{2}$ . The low-lying states, up to level 2, are depicted in Fig. 2 below.

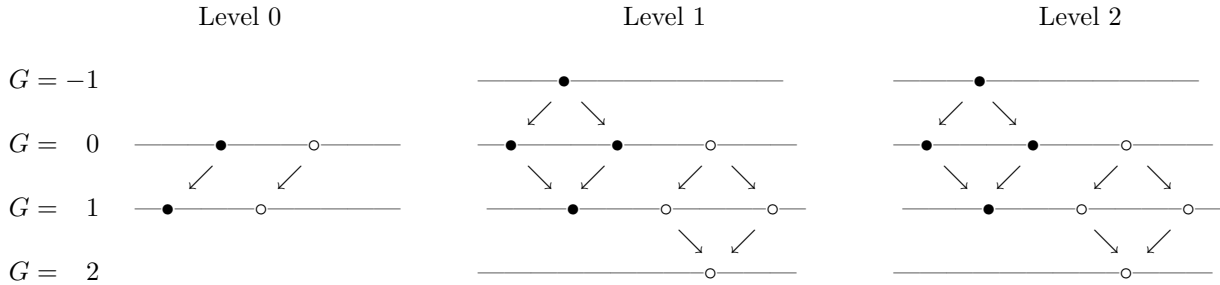


Figure 2

This diagram shows the structure of quartets and their conjugates for each prime state, at levels 0, 1 and 2 in the two-scalar string. Horizontal lines indicate permitted ghost numbers, bullets indicate the quartet states, and circles indicate their conjugates. At level 0 we have a continuum of tachyon prime states; the diamonds degenerate to dumbbells in this case. At level 1, we have the  $SL(2, C)$  vacuum as prime state. At level 2, there are two prime states, corresponding to the  $x$  and  $y$  ring generators.

<sup>†</sup> Note that the quartet structure that we have been describing is equivalent to the “diamond” structure of [4]. However, in our description we use the two BRST non-trivial operators  $a_\varphi$  and  $a_X$ , rather than just the  $a_\varphi$  in [4], giving us explicit generators for all the states in the quartets. States built with  $a_X$  may differ from those in [3,4] by BRST-trivial states. For example, at level  $\ell = 1$  the conjugate state  $|-G\rangle_2$  with  $G = 1$  has the form  $c_{-1}|\vec{p}, -\rangle$ , whereas the corresponding state given in [3] is of the form  $\alpha_{-1}^1|\vec{p}, +\rangle$  (where  $\alpha_n^1$  denotes the Fourier modes of the Liouville field  $\varphi$ ). The two differ by the BRST-trivial state  $Q_{BRST}b_{-1}|\vec{p}, +\rangle$ .

Note that all the prime states discussed so far are manifestly cohomologically non-trivial, since they each have the lowest possible ghost number for their level and so there are no states of which they can be the BRST variations. Conversely, the duals of these prime states, having maximum possible ghost numbers for their levels, are manifestly BRST invariant. All the higher-level prime states correspond to operators that are monomials of  $x$  and  $y$  [3].

As we mentioned in the introduction, in some respects the two-scalar  $W_3$  string resembles the ordinary string with one scalar. Accordingly it is useful to examine the single-scalar string in more detail. The single scalar  $\varphi$  is a Liouville field, with energy-momentum tensor  $T = -\frac{1}{2}(\partial\varphi)^2 - Q\partial^2\varphi$ , where the background charge is given by  $Q^2 = \frac{25}{12}$ . Since now we only have one scalar, there is just one multiplet-generating operator  $a_\varphi = [Q_{BRST}, \varphi] = c\partial\varphi - Q\partial c$ . Hence the multiplets are doublets rather than quartets, at all levels.

For the single-scalar string the tachyon states  $|p, -\rangle = e^{p\varphi(0)}|-\rangle$  become discrete also, with momenta  $p_+ = -\frac{6}{5}Q$  and  $p_- = -\frac{4}{5}Q$ . At level  $\ell = 1$ , there is the  $SL(2, C)$  vacuum, its doublet partner, and their conjugates. At level  $\ell = 2$ , there is now just a single prime state, at  $G = -1$ , given by the operator

$$x = (bc + \frac{3}{5}Q\partial\varphi)e^{\frac{2}{5}Q\varphi} \quad (2.3)$$

acting on the  $SL(2, C)$  vacuum. At this point a significant difference between the one-scalar string and the two-scalar string emerges. For the two-scalar string, the exponential operators in the expressions for the ring generators  $x$  and  $y$  are of the form  $\exp[\frac{1}{\sqrt{2}}(\varphi \pm iX)]$  [3] (see eq. (2.2a-b), with  $Q = \sqrt{2}$ ). Consequently in the OPE of  $x$  with  $x$  or  $y$  with  $y$  the exponentials contribute no  $(z - w)$  factor, and in the OPE of  $x$  with  $y$  they contribute  $(z - w)^{-1}$ . In the one-scalar case, on the other hand, we see from (2.3) that in the OPE of  $x$  with  $x$  the exponentials will contribute a factor of  $(z - w)^{-1/3}$ . It follows that only monomials of the form

$$x^n, \quad \text{for} \quad n = 3p \quad \text{or} \quad n = 3p + 1, \quad (2.4)$$

where  $p$  is a non-negative integer, give operators corresponding to states of well-defined level number  $\ell$ . The level number is given by

$$\ell = \frac{1}{6}(n + 2)(n + 3). \quad (2.5)$$

The first few allowed levels are  $\ell = 1, 2, 5, 7, 12, 15, \dots$ , corresponding to the operators  $1, x, x^3, x^4, x^6, x^7, \dots$ . The operators  $\{x^{3p}\} = 1, x^3, x^6, x^9, x^{12}, \dots$  form a ring, generated by  $x^3$ . The remaining operators  $\{x^{3p+1}\}$  are then obtained as  $x$  multiplied by powers of the ring generator.

An important property of the operator  $x$  in (2.3) is that it maps the tachyon  $|-\frac{6}{5}Q, -\rangle$  into the tachyon  $|-\frac{4}{5}Q, -\rangle$ . Specifically, the normal-ordered product of  $x$  with the  $p = -\frac{6}{5}Q$  tachyon operator gives the  $p = -\frac{4}{5}Q$  tachyon operator. It is by seeking the appropriate generalisation of this property that we find the clue, in the next section, to the level number at which the analogous ring generators arise in the two-scalar  $W_3$  string.

### 3. Discrete states in the $W_3$ string

As we discussed in the introduction, we shall concentrate mainly on the two-scalar realisation of  $W_3$  [5]. The spin-2 and spin-3 primary currents are given by

$$T = -\frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}(\partial\varphi_2)^2 - Q_1\partial^2\varphi_1 - Q_2\partial^2\varphi_2 , \quad (3.1a)$$

$$W = \frac{2i}{3\sqrt{29}} \left[ -\frac{1}{3}(\partial\varphi_1)^3 + \partial\varphi_1(\partial\varphi_2)^2 - Q_1\partial\varphi_1\partial^2\varphi_1 + 2Q_2\partial\varphi_1\partial^2\varphi_2 \right. \\ \left. + Q_1\partial\varphi_2\partial^2\varphi_2 - \frac{1}{3}Q_1^2\partial^3\varphi_1 + Q_1Q_2\partial^3\varphi_2 \right] , \quad (3.1b)$$

where  $Q_1$  and  $Q_2$  are chosen such that

$$Q_1^2 = \frac{49}{8}, \quad Q_2^2 = \frac{49}{24} , \quad (3.2)$$

in order that the central charge take its critical value  $c = 100$ .

The BRST operator has the form

$$Q_{BRST} = \oint dz \left( c(T + \frac{1}{2}T_{\text{gh}}) + \gamma(W + \frac{1}{2}W_{\text{gh}}) \right) . \quad (3.3)$$

The ghost currents  $T_{\text{gh}}$  and  $W_{\text{gh}}$  are given by [6,7]

$$T_{\text{gh}} = -2b\partial c - \partial b c - 3\beta\partial\gamma - 2\partial\beta\gamma , \quad (3.4a)$$

$$W_{\text{gh}} = -\partial\beta c - 3\beta\partial c - \frac{8}{261} [\partial(b\gamma T) + b\partial\gamma T] \\ + \frac{25}{6 \cdot 261} \hbar \left( 2\gamma\partial^3 b + 9\partial\gamma\partial^2 b + 15\partial^2\gamma\partial b + 10\partial^3\gamma b \right) , \quad (3.4b)$$

where the ghost-antighost pairs  $(c, b)$  and  $(\gamma, \beta)$  correspond respectively to the  $T$  and  $W$  generators.

The standard ghost vacuum for the  $W_3$  string is defined by acting on the  $SL(2, C)$  vacuum with  $c_1\gamma_1\gamma_2$  to give

$$|--\rangle \equiv c_1\gamma_1\gamma_2|0\rangle . \quad (3.5)$$

Tachyon states are constructed as  $|\vec{p}, --\rangle \equiv e^{\vec{p} \cdot \vec{\varphi}(0)}|--\rangle$ . General states are obtained by acting on  $|\vec{p}, --\rangle$  with a string of ghost operators  $c_{-m}$ ,  $\gamma_{-m}$ ,  $m \geq 0$ , antighost operators  $b_{-m}$ ,  $\beta_{-m}$ ,  $m \geq 1$ , and matter operators  $\vec{\alpha}_{-m}$ ,  $m \geq 1$ . Thus the ghost number at level  $\ell$  must lie in the interval

$$\left[ 1 - \sqrt{4\ell + 1} \right] \leq G \leq \left[ 1 + \sqrt{4\ell + 1} \right] , \quad (3.6)$$

where  $[x]$  denotes the integer part of  $x$ .

As for the ordinary string, the physical states are defined to be the cohomologically nontrivial states annihilated by  $Q_{BRST}$ . We again have multiplet-generating operators, this time given by  $a_i = [Q_{BRST}, \varphi_i]$ ,  $i = 1, 2$ . They take the form

$$\begin{aligned}
a_1 &= c\partial\varphi_1 - Q_1\partial c + \frac{8}{261}\partial\varphi_1 b\gamma\partial\gamma - \frac{8}{261}Q_1(\partial b\gamma\partial\gamma + b\gamma\partial^2\gamma) \\
&\quad + \frac{2i}{\sqrt{261}}\left((\partial\varphi_1)^2\gamma - Q_1\partial\varphi_1\partial\gamma - (\partial\varphi_2)^2\gamma - 2Q_2\partial^2\varphi_2\gamma + \frac{1}{3}Q_1^2\partial^2\gamma\right) \\
a_2 &= c\partial\varphi_2 - Q_2\partial c + \frac{8}{261}\partial\varphi_2 b\gamma\partial\gamma - \frac{8}{261}Q_2(\partial b\gamma\partial\gamma + b\gamma\partial^2\gamma) \\
&\quad + \frac{2i}{\sqrt{261}}\left(-2\partial\varphi_1\partial\varphi_2\gamma + 2Q_2\partial\varphi_1\partial\gamma + 2Q_2\partial^2\varphi_1\gamma + Q_1\partial\varphi_2\partial\gamma - Q_1Q_2\partial^2\gamma\right).
\end{aligned} \tag{3.7}$$

Acting with  $a_1$ ,  $a_2$  and  $a_1a_2$  on a prime state will generate a quartet of states with ghost numbers  $(G, G+1, G+1, G+2)$  and a quartet of conjugates, which have ghost numbers  $(-G+2, -G+1, -G+1, G)$ . (The conjugate of a state with ghost number  $G$  and momentum  $(p_1, p_2)$  is a state with ghost number  $(-G+2)$  and momentum  $(-p_1 - 2Q_1, -p_2 - 2Q_2)$ .)

We now consider the physical states that are analogous to the level 0, 1 and 2 physical states of the ordinary string. At level zero we again have tachyons. Like the one-scalar string, the tachyons in the two-scalar  $W_3$  string have discrete momenta. This is because we have two constraints and two momentum components. Since these constraints constitute a quadratic polynomial (from  $T$ ) and a cubic polynomial (from  $W$ ), there are  $2 \times 3 = 6$  possible momentum values, which turn out to be:

$$\begin{aligned}
\vec{p}_1 &= (-\frac{6}{7}Q_1, -\frac{6}{7}Q_2), & \vec{p}_2 &= (-\frac{8}{7}Q_1, -\frac{8}{7}Q_2), \\
\vec{p}_3 &= (-\frac{8}{7}Q_1, -\frac{6}{7}Q_2), & \vec{p}_4 &= (-\frac{6}{7}Q_1, -\frac{8}{7}Q_2), \\
\vec{p}_5 &= (-Q_1, -\frac{5}{7}Q_2), & \vec{p}_6 &= (-Q_1, -\frac{9}{7}Q_2).
\end{aligned} \tag{3.8}$$

Note that the pairs of momenta on each line are conjugate to each other. The six states  $|\vec{p}_i, --\rangle$ ,  $i = 1, \dots, 6$  are prime states, which give rise to six quartets *via* the action of the multiplet-generating operators  $a_1$  and  $a_2$ . In fact these quartets, which each span the full range of allowed ghost numbers ( $0 \leq G \leq 2$ ) at level 0, are pairwise conjugate.

In the ordinary string, the next level ( $\ell = 1$ ) has the  $SL(2, C)$  vacuum  $b_{-1}|\vec{0}, -\rangle$  as prime state. For the  $W_3$  string, the  $SL(2, C)$  vacuum is also a prime state, but it now occurs at level  $\ell = 4$ , since it is written in terms of the ghost vacuum  $|\vec{0}, --\rangle$  as  $b_{-1}\beta_{-1}\beta_{-2}|\vec{0}, --\rangle$ . We see from this that it has ghost number  $-3$ , which is the lowest value allowed at level  $\ell = 4$ . It gives rise to a quartet with ghost numbers  $(-3, -2, -2, -1)$ , and a conjugate quartet with ghost numbers  $(5, 4, 4, 3)$  and momentum  $(-2Q_1, -2Q_2)$ .

The question now arises as to the level number at which the analogues of the ring generators  $x$  and  $y$  of the ordinary string will occur. Clearly they should lie at a higher level than that of the  $SL(2, C)$  vacuum, which corresponds to the identity operator in the ring. For the ordinary string, they in fact occurred at one level higher, namely  $\ell = 2$ . For the  $W_3$  string it is not *a priori* obvious whether they should again occur at one level higher than the

$SL(2, C)$  vacuum (*i.e.*  $\ell = 5$ ), or whether on the other hand they should occur at an even higher level. To resolve this, we recall the property noted at the end of section 2 for the operator  $x$  in the one-scalar string. There, we observed that  $x$  maps the tachyon  $|\!-\frac{6}{5}Q, -\rangle$  into the tachyon  $|\!-\frac{4}{5}Q, -\rangle$ .

Since in the present case the tachyons also have discrete momenta,  $\vec{p}_i$  given by (3.8), it is natural to seek operators  $x, y, \dots$  that map one tachyon into another. Thus the requirements for such operators are first of all that they have momenta given by  $\vec{p}_i - \vec{p}_j$  for some tachyon momenta  $\vec{p}_i$  and  $\vec{p}_j$ . Secondly, in order that such an operator  $x$  can act on a tachyon to give another state with well-defined level number, it must be that the OPE of the exponential factor in  $x$  with the tachyon must give an integer power of  $(z - w)$ . Thus we must have

$$(\vec{p}_i - \vec{p}_j) \cdot \vec{p}_j = \text{integer}. \quad (3.9)$$

By enumerating all possible such cases, we find that just two momentum values are possible for the  $x$ -type operators, namely

$$\vec{p} = (\frac{2}{7}Q_1, 0), \quad \text{and} \quad \vec{p} = (\frac{1}{7}Q_1, \frac{3}{7}Q_2). \quad (3.10)$$

Each of these momentum values gives an exponential operator with conformal weight  $\Delta = -\frac{1}{2}\vec{p}^2 - \vec{p} \cdot \vec{Q} = -2$ . The ghost vacuum,  $|\vec{0}, --\rangle \equiv c_1\gamma_1\gamma_2|0\rangle$  has conformal weight  $-4$ , and so for the  $x$  operators to have spin 0 (as they must, in order to map tachyons to tachyons), they must have level  $\ell = 6$ .

We have learnt that the  $x$ -type operators in the  $W_3$  string occur at level  $\ell = 6$ . Of course these operators, which map tachyons to tachyons, must have ghost number  $G = 0$ . This means that the corresponding *states* at level 6 will have ghost number  $G = -3$ . From (3.6), we see that the lowest allowed ghost number at level 6 is  $G = -4$ . Thus we have a rather different situation from the ordinary string, in that the  $x$ -type state does not have the minimum allowed ghost number. This means that when we solve for such states by demanding that they are annihilated by  $Q_{BRST}$ , we must then check that they are not merely BRST variations of some  $G = -4$  states. Also, it raises the possibility that the  $x$ -type states might in principle be non-prime states obtained by acting on a prime state of ghost number  $G = -4$  with  $a_1$  and  $a_2$ . In fact, this turns out not to be the case, as we shall describe below.

The considerations above lead us to look for spin 0,  $G = 0$ , level 6 operators. The most general such operator, at arbitrary momentum  $\vec{p}$ , has 30 parameters  $g_i$  for the prefactors of the exponential, and is given in the Appendix in eq. (A.2). Some representative terms are  $(\partial^2\vec{\varphi}, \partial bc, \partial\beta\gamma, \dots)e^{\vec{p}\cdot\vec{\varphi}}$ . We then require that this operator be annihilated by the BRST operator, given by (3.1-3.4). The calculations are straightforward but somewhat tedious, and are best performed by computer. The condition of  $BRST$  invariance leads to 186 linear, homogeneous equations, with  $\vec{p}$ -dependent coefficients, for the 30 parameters  $g_i$ . At



generic values of momentum  $\vec{p}$ , there is a unique solution, up to overall scale. This can be easily understood, since there is a unique level 6 state with ghost number  $G = -4$ , namely  $b_{-1}b_{-2}\beta_{-1}\beta_{-2}|\vec{p}, --\rangle$ . Acting on this with  $Q_{BRST}$  gives a state that is annihilated by  $Q_{BRST}$ , and must therefore be the one mentioned above. It is, of course, BRST trivial.

At certain special values of the momentum, the 186 equations for the 30 coefficients  $g_i$  turn out to leave two parameters, rather than just one, undetermined. In other words, there are two independent solutions at these particular momentum values. One is merely the specialisation of the generic BRST-trivial state mentioned above, whilst the other is a new, BRST non-trivial state. There are in fact two values of the momentum  $\vec{p}$  for which these non-trivial solutions occur, namely the two values given in (3.10). Thus we have independently arrived at the same two momentum values that were indicated by the tachyon-mapping argument. We shall denote the two operators by  $x$  and  $y$ :

$$\begin{aligned} x &= R_x \exp(\tfrac{2}{7}Q_1\varphi_1), \\ y &= R_y \exp(\tfrac{1}{7}Q_1\varphi_1 + \tfrac{3}{7}Q_2\varphi_2), \end{aligned} \tag{3.11}$$

where the prefactors  $R_x$  and  $R_y$  are given in the Appendix, in eqs (A.2–A.4). The operators (3.11) are the  $W_3$  analogues of the  $x$  and  $y$  operators of the ordinary two-dimensional string.

The fact that  $x$  and  $y$  map certain tachyon states to other tachyon states gives an independent proof that they are BRST non-trivial. This follows from the fact that the tachyon states themselves are BRST non-trivial, and so, for example, if two such states  $|t_1\rangle$  and  $|t_2\rangle$  are related by  $|t_2\rangle = x|t_1\rangle$ , then it cannot be that  $x = \{Q_{BRST}, U\}$  for any operator  $U$ , since then one would have  $|t_2\rangle = Q_{BRST}U|t_1\rangle$ , contradicting the BRST non-triviality of  $|t_2\rangle$ .

We also asserted above that  $x$  and  $y$  themselves correspond to prime states, rather than being obtained by acting with  $a_1$  or  $a_2$  on  $G = -4$  level-6 prime states. We see this by establishing that there are no values of the momentum  $\vec{p}$  for which the (unique)  $G = -4$  state  $b_{-1}b_{-2}\beta_{-1}\beta_{-2}|\vec{p}, --\rangle$  is annihilated by  $Q_{BRST}$ , and hence there are no  $G = -4$  prime states at this level.

Just as for the  $x$  operator in the one-scalar string, here, we cannot consider the set of all monomials  $x^m y^n$  for arbitrary integers  $m$  and  $n$ , since the operator products are not always well defined. The signal for a product's being well defined is that the momentum of the resulting operator should be such as to yield an integer-weight exponential operator. This ensures that the monomial  $x^m y^n$  is single valued, and has a well-defined level number. From (3.11), it follows that the conformal weight of the exponential factor in  $x^m y^n$  is

$$\Delta = -\tfrac{1}{4}(m^2 + n^2 + mn + 7m + 7n), \tag{3.12}$$

and the level number of the corresponding state is

$$\ell = 4 - \Delta = \tfrac{1}{4}(m^2 + n^2 + mn + 7m + 7n + 16). \tag{3.13}$$

The allowed values of  $(m, n)$  such that  $\Delta$  is an integer are

$$\left\{ (0, 0), (0, 1), (1, 0), (1, 2), (2, 1), (2, 2) \right\} \bmod 4 . \quad (3.14)$$

The allowed operators are thus all of the form

$$x^{4p}y^{4q}\{1, x, y, xy^2, x^2y, x^2y^2\} , \quad (3.15)$$

where  $p$  and  $q$  are arbitrary non-negative integers. The operators  $x^{4p}y^{4q}$  form a ring, generated by  $x^4$  and  $y^4$ . There is in fact a larger ring, generated by  $x^4$ ,  $y^4$  and  $x^2y^2$ , but it is not so straightforward to state the rules for generating all the operators in (3.15) in terms of this enlarged ring.

So far, we have identified the tachyon operators, at level  $\ell = 0$ , the identity operator, at  $\ell = 4$ , and the  $x$  and  $y$  analogues of the ring generators, at  $\ell = 6$ . We have a possibility here for a richer structure than the ordinary string, where the tachyon, identity, and ring operators occurred at levels 0, 1 and 2 respectively. In particular, we can now look for BRST non-trivial operators with level numbers in between those of the tachyon, the identity, and the  $x$  and  $y$  operators. We have found that there are six such operators at level  $\ell = 1$ . They take the form

$$(c\gamma \pm \frac{i}{\sqrt{522}}\gamma\partial\gamma)e^{\vec{p}\cdot\vec{\phi}} \quad (3.16)$$

with momenta given by

$$\begin{aligned} \vec{p}_1 &= (-\frac{4}{7}Q_1, -\frac{2}{7}Q_2), & \vec{p}_2 &= (-\frac{4}{7}Q_1, -\frac{12}{7}Q_2) , \\ \vec{p}_3 &= (-\frac{3}{7}Q_1, -\frac{5}{7}Q_2), & \vec{p}_4 &= (-\frac{3}{7}Q_1, -\frac{9}{7}Q_2) , \\ \vec{p}_5 &= (-\frac{6}{7}Q_1, 0), & \vec{p}_6 &= (-\frac{8}{7}Q_1, 0) . \end{aligned} \quad (3.17)$$

In (3.16), the  $+$  sign is chosen for the momenta  $\vec{p}_1$ ,  $\vec{p}_2$  and  $\vec{p}_5$ , and the  $-$  sign for  $\vec{p}_3$ ,  $\vec{p}_4$  and  $\vec{p}_6$ . The six operators correspond to states  $(\beta_{-1} \mp \frac{i}{\sqrt{522}}b_{-1})|\vec{p}, --\rangle$ , which have  $G = -1$ , the lowest allowed ghost number at level 1. These prime states give rise to quartets and conjugates in the usual way.

We have not looked exhaustively at all allowed ghost numbers for levels 2, 3 and 5. However, we have verified that there are no discrete states at the lowest permitted ghost numbers for these levels. The low-lying states are depicted in Fig. 3 below.

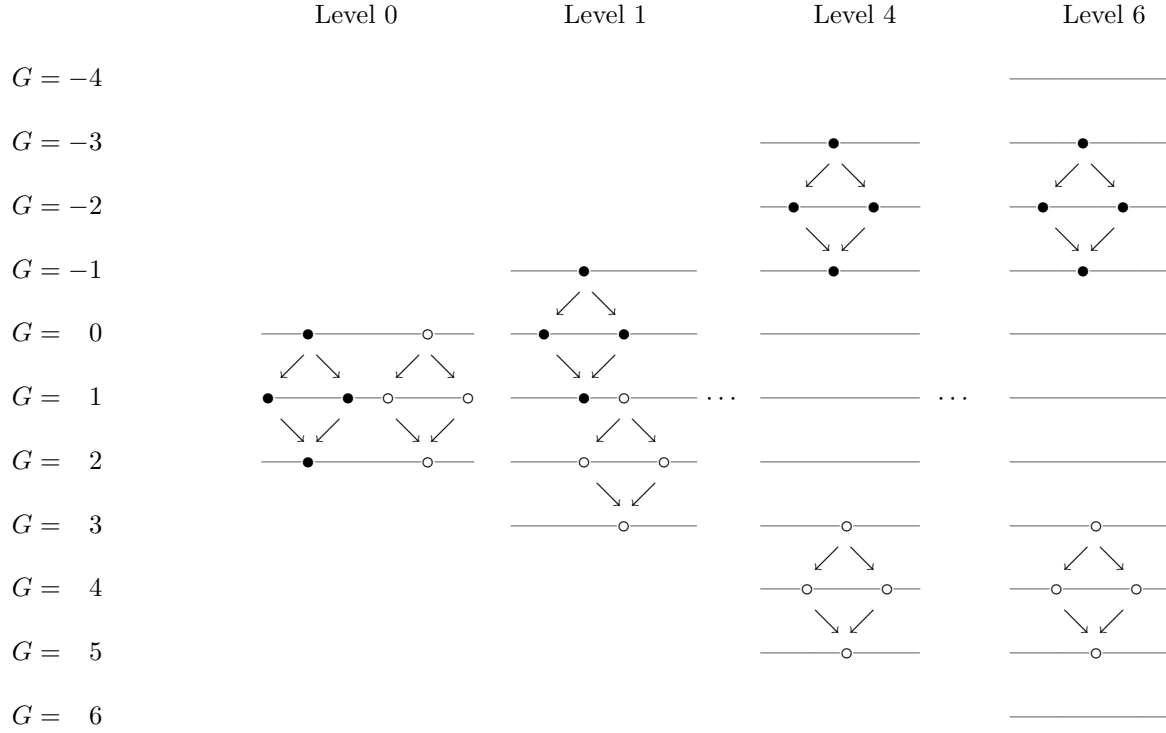


Figure 3

This diagram shows the structure of the quartets and their conjugates for each prime state, at levels 0, 1, 4 and 6 in the two-scalar  $W_3$  string. The notation is analogous to that of Fig. 2. At level 0 there are six tachyon prime states; at level 1 there are six prime states. At level 4 there is one prime state, namely the  $SL(2, C)$  vacuum. At level 6 there are two prime states, corresponding to the  $x$  and  $y$  operators described above.

#### 4. Discussion

For the ordinary string in two dimensions, the ring of  $G = 0$  operators generated by  $x$  and  $y$ , together with their associated quartets and conjugates, comprise the complete set of BRST-nontrivial discrete operators (at least for the chiral sector). For the two-scalar  $W_3$  string, it is not clear whether the states described in the previous section exhaust the BRST non-trivial discrete states. In fact the existence of the level-1 states described above, which have no direct analogues in the ordinary string, suggests that the story may be a more complicated one. As we shall now argue, they may indicate the existence of further  $x$  and  $y$ -type operators at higher level numbers.

Recalling that we deduced the existence, and level number, for the  $x$  and  $y$  operators (3.11) by seeking operators that mapped tachyons into tachyons, we can carry out a similar discussion for hypothetical operators that map the set of level-1 prime states into itself.

In fact  $x$  and  $y$  themselves have this property:  $x$  maps the level-1 state with momentum  $\vec{p}_6$  in (3.16) into the state with momentum  $\vec{p}_5$ , and  $y$  maps the state with momentum  $\vec{p}_2$  into the state with momentum  $\vec{p}_4$ . There are other pairs of level-1 states that satisfy the necessary condition (3.9) for the existence of an operator that could map one into the other. These would correspond to operators at level  $\ell = 8$  with momenta  $(\frac{4}{7}Q_1, -\frac{2}{7}Q_2)$ ,  $(\frac{4}{7}Q_1, -\frac{12}{7}Q_2)$ ,  $(\frac{1}{7}Q_1, Q_2)$  and  $(-\frac{4}{7}Q_1, \frac{12}{7}Q_2)$ ; and operators at level  $\ell = 9$  with momenta  $(0, \frac{10}{7}Q_2)$ ,  $(\frac{5}{7}Q_1, -\frac{5}{7}Q_2)$ ,  $(\frac{5}{7}Q_1, -\frac{9}{7}Q_2)$  and  $(-\frac{2}{7}Q_1, \frac{12}{7}Q_2)$ . These operators, having ghost number 0, would correspond to states with ghost number  $-3$ . The minimum allowed ghost numbers at levels 8 and 9 are  $-4$  and  $-5$  respectively. Thus to check explicitly for the existence of such operators would be quite complicated, and, especially in the  $\ell = 9$  case, would involve recognising and discarding many BRST-trivial states.

It is interesting to note that the six level-1 prime states, with momenta given in (3.17), give rise to 12 physical states with the “standard” ghost number,  $G = 0$ . Those with momenta  $\vec{p}_5$  and  $\vec{p}_6$  in (3.17) correspond to physical states that have been found in earlier work on the spectrum of the  $W_3$  string [8,9]. However, the states with momenta  $\vec{p}_1, \dots, \vec{p}_4$  are of a kind that have not been previously described. They correspond to physical states involving ghost, as well as matter, excitations. It would seem that the spectrum of physical states in the  $W_3$  string is thus richer than had previously been thought.

The two-scalar realisation of  $W_3$  is intimately related to the Lie algebra of  $SU(3)$ . It would not be surprising, therefore, if the discrete states of the  $W_3$  string were to exhibit an  $SU(3)$  structure, possibly with its enveloping algebra arising as a symmetry algebra of the states. This would be a generalisation of the wedge subalgebra of  $w_\infty$ , found as a symmetry of the states of the two-scalar ordinary string [3].

In the two-scalar ordinary string, one can make an association between a physical state  $|\chi\rangle$  and a spin-1 current  $j(z)$ , defined by  $j(0)|0\rangle = b_{-1}|\chi\rangle$  [3]. The current  $j(z)$  defines a charge that automatically commutes with the BRST operator. If one requires that  $j(z)$  be a primary field, then the physical state  $|\chi\rangle$  must be annihilated by  $b_0$ . It is these conserved currents that generate the symmetry algebra described above. For the two-scalar  $W_3$  string, we can again build spin-1 currents from physical states, by acting with  $b_{-1}$  on physical states. These again give charges that commute with the BRST operator. If we require that the currents must be primary fields under the energy-momentum tensor, then this imposes the restriction that the physical states must be annihilated by  $b_0$  and  $\beta_0$ . In other words, these conditions on the physical state  $|\chi\rangle$  ensure that  $b_{-1}|\chi\rangle$  is a highest-weight state with respect to the Virasoro algebra. In fact, they also ensure that  $b_{-1}|\chi\rangle$  is annihilated by the Laurent modes  $W_n$  of the spin-3 current, for  $n > 0$ . However, there seems to be no way of having the state  $b_{-1}|\chi\rangle$  also be an eigenstate of  $W_0$ . Whether this presents a problem for interpreting the conserved charges as generators of a symmetry algebra is not clear to us. We note from (A.2-A.4) in the appendix that for the  $x$  and  $y$  operators at level 6, the corresponding states will indeed be annihilated by  $b_0$ , since  $g_{11}$  vanishes, and that in each

case there exists a choice of the parameters  $\lambda$  and  $\tau$  such that the states are annihilated by  $\beta_0$  as well, by making the coefficient  $g_{12}$  vanish.

One might think that a three-scalar, rather than two-scalar, realisation of  $W_3$  would provide a more natural generalisation of the two-scalar ordinary string. In particular, the tachyons would then have continuous momenta in the  $\varphi_2$  and  $\varphi_3$  directions (the unfrozen “spacetime” dimensions in the terminology of [7,8,9]). We have looked at the conditions for the existence of level-6  $x$ - and  $y$ -type operators in this case, and found that there is now only one BRST non-trivial example, with momentum  $(\frac{2}{7}Q_1, 0, 0)$ . At level 1, we find that of the six states given in (3.17), the last two generalise to discrete states with momenta  $(-\frac{6}{7}Q_1, 0, 0)$  and  $(-\frac{8}{7}Q_1, 0, 0)$ , whilst the first four generalise to states with continuous “spacetime” momenta,  $(-\frac{3}{7}Q_1, p_2, p_3)$  and  $(-\frac{4}{7}Q_1, p_2, p_3)$ . In fact the phenomenon of continuous-momentum states repeats at higher levels. It can be understood from the fact that the effective spacetime theory for a  $W$  string has fewer gauge symmetries than an ordinary string, where two out of the spacetime dimensions describe unphysical longitudinal modes.

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## APPENDIX

In this appendix, we present the details of the discrete level-6 operators  $x$  and  $y$  in the two-scalar  $W_3$  string. They correspond to states at ghost number  $G = -3$ , and can be written as

$$R e^{\vec{p} \cdot \vec{\varphi}}, \quad (A.1)$$

where the most general possible prefactor  $R$  at this level and ghost number takes the form

$$\begin{aligned} R = & g_1 b b' \gamma \gamma' + g_2 \beta c + g_3 b \beta \gamma \gamma' + g_4 b' c + g_5 \beta \gamma' + g_6 b b' c \gamma \\ & + g_7 b' \gamma' + g_8 b'' \gamma + g_9 \beta' \gamma + g_{10} b \beta c \gamma + g_{11} b c' + g_{12} b \gamma'' \\ & + g_{13} \varphi'_1 b \gamma' + g_{14} \varphi'_2 b \gamma' + g_{15} \varphi'_1 b' \gamma + g_{16} \varphi'_2 b' \gamma + g_{17} \varphi'_1 b c + g_{18} \varphi'_2 b c \\ & + g_{19} \varphi'_1 \beta \gamma + g_{20} \varphi'_2 \beta \gamma + g_{21} \varphi''_1 b \gamma + g_{22} \varphi''_2 b \gamma + g_{23} \varphi''_1 + g_{24} \varphi''_2 \\ & + g_{25} (\varphi'_1)^2 + g_{26} \varphi'_1 \varphi'_2 + g_{27} (\varphi'_2)^2 + g_{28} (\varphi'_1)^2 b \gamma + g_{29} \varphi'_1 \varphi'_2 b \gamma + g_{30} (\varphi'_2)^2 b \gamma, \end{aligned} \quad (A.2)$$

with  $'$  denoting the holomorphic derivative  $\partial$ .

As described in section 3, for general values of the momentum  $\vec{p}$ , there is a unique solution, up to overall scale, for the coefficients  $g_i$ , following from the physical-state requirement  $Q_{BRST} R e^{\vec{p} \cdot \vec{\varphi}(0)} |0\rangle = 0$ . This corresponds to the BRST-trivial state obtained by acting with  $Q_{BRST}$  on the ghost-number  $G = -4$  state  $b_{-1} b_{-2} \beta_{-1} \beta_{-2} |\vec{p}, --\rangle$ . At each of the two special values of momentum  $\vec{p} = (\frac{2}{7}Q_1, 0)$  and  $\vec{p} = (\frac{1}{7}Q_1, \frac{3}{7}Q_2)$ , a second solution occurs; this is BRST non-trivial. The two-parameter family of solutions for  $\vec{p} = (\frac{2}{7}Q_1, 0)$  is:

$$\begin{aligned} g_1 &= \frac{4i}{3\sqrt{29}}(63\lambda - 65\tau), & g_2 &= -1566\sqrt{2}(\lambda + \tau), & g_3 &= -60\sqrt{2}(\lambda + \tau), \\ g_4 &= 6i\sqrt{29}(-9\lambda + 7\tau), & g_5 &= 18i\sqrt{29}(-7\lambda + 9\tau), & g_6 &= 12\sqrt{2}(\lambda + \tau), \\ g_7 &= -97\sqrt{2}(\lambda + \tau), & g_8 &= -32\sqrt{2}(\lambda + \tau), & g_9 &= 48i\sqrt{29}(-3\lambda + \tau), \\ g_{10} &= 72i\sqrt{29}(\lambda + \tau), & g_{11} &= 0, & g_{12} &= -8\sqrt{2}(7\lambda + 19\tau), \\ g_{13} &= -88\lambda + 56\tau, & g_{14} &= 0, & g_{15} &= 8(\lambda + \tau), \\ g_{16} &= 0, & g_{17} &= -24i\sqrt{58}(3\lambda + \tau), & g_{18} &= 0, \\ g_{19} &= -72i\sqrt{58}(\lambda + \tau), & g_{20} &= 0, & g_{21} &= 32(2\lambda + 3\tau), \\ g_{22} &= \frac{224}{\sqrt{3}}\tau, & g_{23} &= 24i\sqrt{58}(-4\lambda + \tau), & g_{24} &= 56i\sqrt{174}\tau, \\ g_{25} &= -48i\sqrt{29}(2\lambda + \tau), & g_{26} &= 0, & g_{27} &= 48i\sqrt{29}\tau, \\ g_{28} &= 32\sqrt{2}\lambda, & g_{29} &= 0, & g_{30} &= 32\sqrt{2}\tau, \end{aligned} \quad (A.3)$$

where  $\lambda$  and  $\tau$  are arbitrary parameters. The BRST trivial operator corresponds to the choice  $\lambda = -\tau$ . For the momentum  $\vec{p} = (\frac{1}{7}Q_1, \frac{3}{7}Q_2)$ , the two-parameter family of solutions is:

$$\begin{aligned}
g_1 &= \frac{4i}{3\sqrt{29}}(129\lambda - 127\tau) , & g_2 &= -1566\sqrt{2}(\lambda + \tau) , & g_3 &= -60\sqrt{2}(\lambda + \tau) , \\
g_4 &= 6i\sqrt{29}(-15\lambda + 17\tau) , & g_5 &= 18i\sqrt{29}(-17\lambda + 15\tau) , & g_6 &= 12\sqrt{2}(\lambda + \tau) , \\
g_7 &= -97\sqrt{2}(\lambda + \tau) , & g_8 &= -32\sqrt{2}(\lambda + \tau) , & g_9 &= 48i\sqrt{29}(-3\lambda + 5\tau) , \\
g_{10} &= -72i\sqrt{29}(\lambda + \tau) , & g_{11} &= 0 , & g_{12} &= -8\sqrt{2}(25\lambda + \tau) , \\
g_{13} &= 64\lambda - 80\tau , & g_{14} &= 16\sqrt{3}(4\lambda - 5\tau) , & g_{15} &= 4(\lambda + \tau) , \\
g_{16} &= 4\sqrt{3}(\lambda + \tau) , & g_{17} &= 48i\sqrt{58}\tau , & g_{18} &= 48i\sqrt{174}\tau , \\
g_{19} &= 36i\sqrt{58}(\lambda + \tau) , & g_{20} &= 36i\sqrt{174}(\lambda + \tau) , & g_{21} &= 32(7\lambda - \tau) , \\
g_{22} &= \frac{128}{\sqrt{3}}\tau , & g_{23} &= 24i\sqrt{58}(-7\lambda + 5\tau) , & g_{24} &= 64i\sqrt{174}\tau , \\
g_{25} &= 48i\sqrt{29}(-\lambda + \tau) , & g_{26} &= 48i\sqrt{87}(\lambda + \tau) , & g_{27} &= 96i\sqrt{29}\tau , \\
g_{28} &= 32\sqrt{2}\lambda , & g_{29} &= 32\sqrt{6}(-\lambda + \tau) , & g_{30} &= 32\sqrt{2}\tau .
\end{aligned} \tag{A.4}$$

The BRST-trivial solution again corresponds to  $\lambda = -\tau$ . Note that for both cases, the coefficient  $g_{11}$  vanishes. From (A.2), we see that this implies that the corresponding states  $x(0)|0\rangle$  and  $y(0)|0\rangle$  are annihilated by  $b_0$ . By adjusting the parameters so that the coefficient  $g_{12}$  vanishes too, we obtain states that are annihilated by  $\beta_0$  as well. As discussed in section 4, these states may be interpreted as giving rise to conserved currents.

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